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Relationships between area and perimeter: beliefs of teachers

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Work carried out within the framework of the Research Programme of the University of Bologna (Mathematics Department): «Methodological aspects (theoretical and empirical) of the initial and in-service training of mathematics teachers at every scholastic level. ».

Summary. *In this paper we examine teachers' beliefs connected to the relations that exist between the area and perimeter of a plane figure. The research joins, with many new features, a classic mainstream that has been explored considerably for over 60 years. In particular we examine the change of beliefs, the language used to express them and the degree of incidence of the examples we provide. We discuss an idea according to which the supposed relations between area and perimeter are an example of the student's behaviour that leads him to confirm, without criticism, increases and reductions between entities that are placed in relationship.*

1. Introduction and theoretical table

The critical reflections on the problem of learning the concepts of perimeter and area of plane figures can boast of the fact of having been amongst the first to be studied. After concerning himself with the birth of thought and of language in infants, then with the acquisition – construction of the idea of number (with its various meanings), Piaget concerned himself, starting from the 1930's, with conceptual

constructions having to do with Geometry. Amongst the various works which it would be possible to cite here, we limit ourselves to those in which perimeter and area specifically appear or where there are references to such concepts (Piaget, 1926; Piaget, 1937; Piaget, Inhelder, Szeminska, 1948; Piaget, Inhelder, 1962). In the 50's and 60's, these basic works were quickly followed by studies of pupils or followers of the school master from Geneva, based on the same certainties treated by genetic epistemology, for example, Vihn et al. (1964), Vihn, Lunzer (1965). We also mention Battro's study (1969) which repeats all of the celebrated experiments of the master.
These are the studies that have, for over 20 years, conditioned the successive analyses on the same theme. They were based, overall, on the failures of the young pupils at determined stages – ages. In particular, in this vein, the ideas of length and area, amongst others, were studied with great attention, highlighting the great difficulty on the part of the pupils to appropriate the idea of surface. Even more specifically, with the changing of the shape, the research highlighted how the young student tended not to be able to accept the invariance of the surface measurement. The difficulties tied to the false relationship between area and perimeter seem to continue up to the age of 12, according to this research, and they are even more connected to the linguistic development of the subject.
[It is well noted that Piaget's conclusions were subjected to severe criticism on the part of later scholars; so as not to make this work heavier, we refer back only to Resnick, Ford (1981, above all chap. 7)].
Following these preliminary and classic studies, abundant other research was done, so much so that it is impossible here and now to give the complete picture. We will limit ourselves (following a chronological path) only to those that, in some way, refer to the difficulty specific to the learning of the ideas of perimeter and area. These have, without doubt, conditioned the direction of our current research.
A very detailed analysis of the frame relative to previous researches into area and perimeter can be found in D'Amore, Fandiño (2006b): we invite the reader to refer to that article, to shorten this one.

2. Research problems

It is evident, therefore, that the two geometric concepts: *perimeter / area of a plane figure*, have many common elements at the scientific level, but also many others that are simply supposed at the level of misconceptions; very diffuse amongst the students at every scholastic level.
The literature has amply shown (for example, see Stavy, Tirosh, 2001, and many of the above cited articles) how many students of every age are convinced that there is a close dependence relationship between the two concepts on the relational plane, of the type:
if A and B are two plane figures, then:

- if (perimeter of A > perimeter of B) then (area of A > area of B);
- likewise with <;
- likewise with = (for which: two isoperimetric figures are necessarily equi-extensive);

and vice versa, exchanging the order "perimeter – area" with "area – perimeter".
Rarely is this theme taken into didactic examination in an explicit way, also because of a supposed difficulty, according to the teachers.
We might wonder if on the teachers part, at any scholastic level, there is full awareness of the theme or if, by chance, also for some teachers there are problems of conceptual construction. This obviously concerns the problems of the beliefs and conceptions of the teachers.[1]

If we place the perimeters of two figures A and B into relationship, with their respective areas, it seems to us that a convincing way to highlight that the "laws" mentioned above are NOT valid is:
to show an example for each of the following 9 possible cases:

p	**S**	**p**	**S**	**p**	**S**

[1] We deem it necessary to state explicitly that we use the following interpretations of such terms (also proposed at the opening of: D'Amore, Fandiño Pinilla, 2004), always, however, more diffuse and shared:

- *belief* (or credence): opinion, set of judgments/expectations, that which one thinks about something;
- the set of beliefs of someone (A) about something (T) gives the *conception* (K) of A relative to T; if A belongs to a social group (S) and shares with others belonging to S that set of convictions relative to T, then K is the conception of S relative to T.

Often, in place of "conception of A relative to T" one speaks of "the image that A has of T".

>	>	>	=	>	<
=	>	=	=	=	<
<	>	<	=	<	<

The first box **> >** says:

- find two figures such that, passing from the first to the second, the perimeter grows and the area grows

and so on.

To avoid difficulties, one can always give simple figures, such as a rectangle, when it is possible, carrying out the various transformations on it or on figures derived from it. It seems necessary to us to make sure that the figures treated are the most elementary possible, that is, the most usual figures that are found in textbooks and in classrooms, to avoid complications owing to the figure itself.

In the Appendix, the 9 examples as above are given in extremely elementary cases. These examples are never supplied firstly to the subjects who submit to the test that will be described later. Each of the subjects, on his own, must take care of finding appropriate examples, at least in the first instance.

3. Questions, research methodology and hypotheses of answer[2]

To a group of collaborators,[3] primary school teachers, middle school, upper school and university, we proposed, making themselves responsible for the above mentioned research, giving the following indications which are contemporaneously, the explicit demands of the research, the relative methodological indications and our hypotheses of

[2] Contrary to our usual practice, in this work we do not separate these three points because this time they are profoundly connected to each other.

[3] For "collaborators" we mean as follows: in Italy there are research groups in the didactics of mathematics recognised by the Ministry of the University and of Research at the different university seats. The Bologna group is called RSDDM (www.dm.unibo.it/rsddm). As part of these groups there are also teachers at every scholastic level who are officially called "research collaborators". In the group of collaborators, there are also university teachers or others at the same level.

response, subdivided into 3 points. We decided to do the research at all levels of instruction to verify if the results could have had to do, in a specific way, with a scholastic level or instead if the results could be encountered independent of the "scholastic level" variable.

POINT 1.

Research PROBLEM R1: We asked all the collaborators to put themselves to the test, in complete sincerity, and some of their colleagues in the primary, middle, upper schools, as well as university students in teacher training. The problem consists in checking, first in the collaborators themselves and then in the other subjects of the trial, if a change of beliefs happens relative to the relationship between area and perimeter.

Research QUESTION Q1: Is it true or false that one can find examples for all 9 cases? Is it true or false that it is spontaneous to think that at the increasing of the perimeter of a plane figure the area of it is increased, in general? Is it true or false that it is necessary to turn to the cognitive and to ones own experience, to *convince oneself* that things are not like this?

Answer HYPOTHESES H1: We believed that not only for many students, but also for some teachers and some collaborators there were deep-rooted misconceptions with regards to supposed necessary relationships between perimeters and areas of plane figures. That it was not so trivial to find the 9 mentioned examples (especially in the case in which the perimeter must decrease and the surface increase and vice versa). That even after having seen these examples, there was some resistance. As indicators of these deep-rooted misconceptions we thought we would take on the declarations of the same collaborators, of the teachers interviewed and of the university students.

POINT 2.

Research PROBLEM R2: We asked all of the collaborators to do some tests on the primary, middle, upper and university students (from every kind of faculty) and not only students in training as future teachers. Each of these was invited to introduce, in a conversational oral form[4], any discussion they wanted on the perimeter and area of a simple plane figure and try to carry out the transformations, verifying if the students

[4] This introduction to the theme was the proposal of the so-called "Galilean problem of town squares" that we will see shortly.

- accept spontaneously
- accept willingly after an example
- accept with difficulty after several examples
- …
- reject without discussion
- reject even after examples
- …

that all 9 relationships can be valid and that NOTHING can be said a priori about the ties between "increase (equality, decrease) of the perimeter" and "increase (equality, decrease) of the area of plane figures". Each collaborator had to speak to the students, as the first thing proposing the problem, listening to the first answer, making a note of it according to the said scale. As the second, proposing the 9 tests and assisting the students in their execution, listening to their comments. As the third, reaching an explicit formulation of his new belief, in case there is one.

We were interested in observing two things:

a) the *change of beliefs*; if, that is, after some examples, the students are willing to change belief, according to the hypotheses of the theoretical framework presented before, and if age has an influence on this; it thus became essential to have the subjects express their beliefs *before* and *after* the examples. To reach this aim, more than just doing the tests, it became essential to interview the subjects in small groups (2-3 per group) or individually;

b) the language that the students use to explain their thought, before and after: examples, general discussions, sentences, …, use of designs, of diagrams, …

Research QUESTION Q2: With how much naturalness and spontaneity do the students manage to accept that there do not exist obligatory relationships between the perimeter and area of plane figures? How does this acceptance vary with age? Does it turn out to be easy to accept the 9 examples? How do they express their beliefs? What kind of language do they use?

Answer HYPOTHESES H2: We believed that the students, at any age, would express great difficulty in accepting that which seemed anti-intuitive. That is, we thought that more than one student was convinced, before the test, that at the increase of the perimeter there *necessarily* corresponded an increase of the area, for example, and that the more he considered this was obvious and intuitive, the more he would have stated his own effort to accept the result of the test itself. On the basis of our research experience, we considered that with the increase in age this acceptance has a net rise. That the subjects would find some difficulty in accepting the examples. That they would have expressed their beliefs in a minimally academic way, given that they contrast with the scholastically built. That the language used would have tended to be the most colloquial possible, perhaps with the spontaneous use of graphics and schematic designs.

In all, we had 14 *collaborators*:
7 primary school teachers
2 middle school teachers
3 upper school teachers
2 teachers from the university (or equivalent).

Each of these put himself and some colleagues to the test. In all, the number of *teachers* who underwent the test were:
26 from the primary school
16 from the middle school
13 from the upper school
2 teachers from the university,
for a total of 57 teachers.

It should be remembered that all the tests were carried out in the form of individual interviews, of a clinical kind, using the grid described in paragraph 3 of point 2. On the other hand, the interview, often in the cases in which the interviewee was one of the collaborators, was often a kind of *personal story*, given that we tried to have answers which revealed the changes of conscious beliefs. In this research we have made broad use of the written statements of the teachers involved. With various names, this technique has been profitably used a lot in an international context for a while.

4. Research results, discussion of the results and answers to the research questions

4.1.
As regards point 1., research problem R1, we made a distinction between the two handouts:

- we asked all the collaborators to put themselves to the test

and

- some colleagues from the primary, middle and upper schools, as well as students of university courses (specialisation courses in Italy and Master's courses in Switzerland) for the training of secondary school teachers (lower and upper).

In both cases the research questions Q1 were the following: Is it true or false that one can find examples for all 9 cases? Is it true or false that it is spontaneous to think that at the increase of the perimeter of a plane figure, the area of it increases, in general? Is it true or false that it is necessary to strain to *convince oneself* that things are not like this?,
while our answer hypotheses H1 were: we believed that on the part of some teachers (including some collaborators) there were deep-rooted misconceptions about supposed necessary relationships between perimeters and areas of plane figures. That it wasn't so banal to find the said 9 examples [especially in the case of (p<, S>) in which the perimeter must decrease and the surface increase]. That even after having seen the examples, there was some resistance.

In this paragraph **4.1.** we will examine the case in which the subjects who underwent the (self)test were our same collaborators in the research, while we put off to paragraph **4.2.** the case in which the subjects that underwent the test were colleagues of our research collaborators or university students from the faculties mentioned previously.

We have rather similar reactions from the 14 research collaborators as regards the modality of response:

- 1 subject (university teacher) limited himself to carrying out an exclusively mathematical analysis of the question, obviously correct, not responding to the personal question about his own difficulties;

- 13 wrote texts that go from 1 to 6 pages in response, sometimes rather rich with references to their own difficulties:
    - 9 collaborators (7 primary teachers, 1 upper, 1 university) confess their own difficulty at the moment of having to give form to their ideas, even if correct and conscious. They also admit that they had to force themselves to imagine all 9 situations;
    - 4 collaborators (2 middle teachers, 2 upper) stated that they had no problem immediately finding the answers and above all they stated their full awareness that the things had to work in that way.

[4 collaborators (2 primary, 2 middle) make full reference to their own pupils, not managing to answer in the first person only as subjects, but interpreting our question as an implicit invitation to think of a classroom situation].

The case that was declared almost unanimously as the most difficult was exactly that one (p<, S>) which we had supposed and its analogous one (p>, S<).

Our hypotheses H1 are therefore fully confirmed; even by people with a high level of education, like our collaborators, there are, at least at first sight, deep-rooted misconceptions about the supposed necessary relationships between perimeters and areas of plane figures. As indicators of such misconceptions, we had decided to take on either their own explicit admissions or the evident proof of their difficulties. For many, it wasn't so banal to find the 9 examples mentioned [especially in the cases (p<, S>) and, a bit less, (p>, S<)], by their explicit admission. One of the collaborators stated explicitly in writing «(…) I had greater difficulties finding figures in the cases where the perimeter had to decrease and the area had to remain the same or increase», a sentence that we take as a prototype for many others of the same tenor.

One can well see how the self-declarations of difficulty are more numerous amongst the teachers of the first scholastic levels, perhaps because of the lower technical preparation (reported by more than one; many primary school teacher collaborators confess to having learned how to critically treat these questions within the framework of courses organised by the NRD of Bologna).

The choice of the figures for the 9 cases is more numerous, at least at the beginning, around convex polygons and specifically rectangles.[5]

4.2.

The 43 teachers interviewed (19 from the primary school, 8 from the middle, 10 from the upper, 6 in post graduate training as lower secondary school teachers) had very dissimilar behaviours, but also many reactions in common. The protocols of the interviews are available; here we will pick only the essential. We will report between «» the sentences that confirm our affirmations and that seem most representative.

A very diffuse reaction, at all scholastic levels, is the difference shown at the intuitive level upon the first contact with the problem as regards the change (sometimes strong) between the first intuitive response and the belief acquired at the end of the test.

As we said at the beginning, almost every interview began with the so-called "problem of Galileo": «A town has two squares A and B; the perimeter of square A is greater than the perimeter of square B; which of the two squares has the greater area?».[6]

Very many of those interviewed, decidedly the great majority, 40 out of 43, even university graduates and upper school teachers, affirmed that the square that has the greater area is that with the greater perimeter, except for then:

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[5] A note, only as a curiosity, outside of the research. One of the collaborators stated having put some of his family members to the test:

- those involved in construction activities, who daily confront situations in which the cases (p>, S<) and (p<, S>) are recurrent, did not have problems, not only responding correctly, but also supplying examples;
- others, involved in more routine activities, showed that they tended to give the expected classical answers: there exist only the cases (p>, S>), (p<, S<), (p=, S=); the other cases are believed to be impossible: for example, it was not believed possible to find examples for the case (p<, S>).

[6] About the exact history of this problem as it is encountered really in Galileo, one can see D'Amore, Fandiño Pinilla (2006a).

- spontaneously correcting oneself, affirming that "it isn't necessarily so", even before carrying out all the tests foreseen in the interview (and here one notes a greater gathering amongst the upper school teachers)

or

- accepting that their own answer might be criticisable or incorrect, but only after having carried out the tests (and here one notes a greater gathering amongst the teachers at the first scholastic levels).

Therefore, the *change of belief* is obvious, sometimes strong, and in many cases requires proofs and not insignificant reflection.

To the questions: «Is it true or false that one can find examples for all 9 cases? Is it true or false that it is spontaneous to think that at the increase of the perimeter of a plane figure, the area of it increases, in general? Is it true or false that it is necessary to strain to *convince oneself* that things are not like this?»,
many of the teachers, and NOT necessarily only of the primary school, begin with a 'no' answer, which reveals that the deep-rooted misconceptions with regard to supposed necessary relationships between perimeters and areas of plane figures do not lie only with *some* teachers, as we believed, but with most of them.
For many of those interviewed finding the 9 mentioned examples was not the slightest bit trivial [especially in the case (p<, S>) or vice versa). We had many cases of teachers (even upper school and middle school) who found it necessary to take recourse to the (or some of the) examples supplied by the interviewer. [Many noted the symmetry of the requests and some showed intolerance in the case p=, S= for not simply wanting to apply an isometry or to leave the identical figures].

That which one infers, however, is that, after having seen the examples, either created by the person interviewed himself or proposed by the interviewer, (almost) all the misconceptions connected to intuition disappeared. One arrives at sentences full of awareness, such as the following: «Therefore, two equally extensive figures are not automatically also isoperimetric» [this perfect enunciation, was done with obvious surprise by a primary school teacher who stated having struggled a lot with himself to find the 9 examples, stopped by his own beliefs about it, a deep-rooted misconception which before he had never accounted for, that at the increase of the perimeter it might be necessary that the area increase also].

It appears very clear that the misconceptions revealed are due to the fact that almost all the figurative models that accompany these questions are realised with quite usual convex plane figures, which drive one to believe that it is possible to confront the problem ONLY with such figures. Better still, this consideration is confirmed by more than one of the same people interviewed: «It is possible beginning from a square; it is not possible beginning from a circle» (in other words, the square is considered an admissible figure for transformations such as those proposed by us, the circle no); to the proposal of a concave figure: «But this is not a geometric figure» [meaning to say: not of those usually used in didactic practice when one speaks of area and perimeter]; others consider possible only homothety, so: «…but with squares it is impossible» given that the homothetic of a square is still a square.

Very recurrent is the cross-reference that the teachers interviewed made to their own pupils; many of the questions and answers were in fact "filtered" through the experience with or of their own pupils: «They also do no see it» [that which I did not see]; «…they find it hard to imagine it»; «It is necessary to change many figures» [that is, pass from one standard figure to another, for example concave; in reality, it would not always be necessary, but the examples supplied by the interviewers (see Appendix) often are considered as unique].

The fact is interesting that some secondary school teachers (lower and upper) consider this kind of question to be closer to the world of the primary school, «because there one works with figures, more on the concrete, less on the abstract», almost to justify their own failure (and the potential failure of their pupils) in the task. Naturally, there is much truth in this; in the primary school, all to often images that should remain only partial become transformed into deep-rooted models. Often there is not even an awareness of the problem.

The obstacle which will seem evident with respect to the construction of a satisfying mathematical awareness on the relationships between "perimeter and area" is not only of an epistemological nature, but *rather much more* of a didactic nature.

The epistemological nature is obvious and has multiple aspects:
a) it is not by chance that stories and legends which connect area and perimeter are extremely old and are repeated in time, even at the distance of centuries (suffice it to think of the myth of the foundation of Carthage on the part of Dido and the celebrated riddle of Galileo). This is a sign, not more than a sign, obviously, of an epistemological obstacle; on the other hand, when a mathematical idea does not enter immediately as a part of universally accepted mathematics and is, on the contrary, the cause of arguments, contrasts, fights it can generally be considered as an epistemological obstacle in Brousseau's sense (1976-1983; 1986, 1989);
b) to complete these analyses, geometric transformations must be done on the figures; well, only at the end of the 19th century were these transformations, their power, their necessity, revealed to the eyes of the mathematicians. For millennia the staticity of the *Elements* of Euclid dominated. Even this delay in the introduction-acceptance is an obvious sign of an epistemological obstacle.
On the other hand, however, to these obvious epistemological obstacles are also grafted didactic obstacles. If rather profound, appropriate interviews were necessary to change the beliefs of the teachers themselves, how can one not think that their didactic choices used in the classroom with their own pupils don't influence the formation of misconceptions relative to this strategic theme?

5. Conclusions and didactic notes

Seeing the progression of the research with the students, the obstacle which presents itself to the construction of a sufficient knowledge of the relationships between "perimeter and area" is not only epistemological, as is stated in many previous works on this field of research, but rather also of a didactic nature.
It therefore rests in the didactic choices:

- one always uses only convex figures causing the misconception that concave figures *cannot be used* or that using them is unacceptable;
- one always uses only standard figures, causing the misconception that is often expressed with the sentence: «But this is not a geometric figure»;

- almost never are the area and perimeter of the same figure explicitly placed in relationship. In fact, sometimes one insists on the fact that the perimeter is measured in metres (m) while the area is measured in square metres (m^2), insisting on the differences and never on the reciprocal relationships;
- almost never are transformations done on the figures in such a way to preserve or modify the area and perimeter; creating a misconception about the meaning of the term "transformation". Many students, in fact, spontaneously interpret "transformation" to mean a change that only consists in a reduction or an enlargement of the figure (a homothety or a similitude). In the case (p=, S=) many students, as a consequence, refuse the identity or an isometrics as "transformation".

The confirmation of the above also derives from the research carried out on the teachers. There happens, not only in the primary school, the case of the teachers who have reactions which are analogous to those of the students, that is one of surprise when confronted by a necessary change of beliefs. One teacher states: «But if no-one has ever taught us these things, how can we possibly know them?». This seems to us the confirmation of the fact that almost everything can be taken back to didactic obstacles.
The teachers' choices do NOT happen within a correct didactic transposition which lets them act transforming "Knowing" (which for some of them actually there isn't) into a "knowing how to teach", in a learned and aware way (often, unfortunately, there is not even an awareness of the difference between "Knowing" and "knowing how to teach"). Actually, at least in the field investigated by us, a scenario of a-critical questions, hashed and rehashed, is perpetrated following a pre-established script and consecrated by the textbooks. The confirmation is in the following facts: when the teacher changes belief, he does so

- insisting on the fact that this subject should explicitly enter into didactics
- sometimes spontaneously promising himself again to include it in his own didactic action of teaching/learning.

These last considerations allow us to insert the final result of our research on the teachers' change of belief in an important international context. It is true that beliefs can have deleterious effects on didactic action, but the opposite can also be valuable, as our case shows. We are encouraged in this also by the following affirmation: «Beliefs can be an obstacle, but also a powerful force that allows carrying out changes in teaching» (Tirosh, Graeber, 2003).

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APPENDIX

p	S	p	S	p	S
>	>	>	=	>	<
=	>	=	=	=	<
<	>	<	=	<	<